An Eigenvalue Problem for a Fermi System and Lie Algebras

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Abstract We study a Fermi Hamilton operator $\hat{K}$ which does not commute with the number operator $\hat{N}$. The eigenvalue problem and the Schrödinger equation is solved. Entanglement is also discussed. Furthermore the Lie algebra generated by the two terms of the Hamilton operator is derived and the Lie algebra generated by the Hamilton operator and the number operator is also classified.

1 Introduction

In quantum theory Hamilton operators with Fermi-interactions have a long history [1, 2, 3, 4, 5, 6, 7, 8, 9]. Let $c_j^\dagger$, c_j ($j = 1, \ldots, n$) be (spin-less) Fermi creation and annihilation operators, i.e.

$$[c_j^\dagger, c_k]_+ = \delta_{jk} I, \quad [c_j, c_k]_+ = 0, \quad [c_j^\dagger, c_k^\dagger]_+ = 0$$

where $[\,,\,]_+$ denotes the anticommutator and I is the identity operator. Let $|0\rangle$ be the vacuum state. Then $c_j|0\rangle = 0$ and $\langle 0|0\rangle = 1$. Here we study the self-adjoint Hamilton operator

$$\hat{K} = \frac{\hat{H}}{\hbar\omega} = c_n^\dagger c_{n-1}^\dagger \cdots c_2^\dagger c_1^\dagger + c_1 c_2 \cdots c_{n-1} c_n \,.$$

The number operator $\hat{N}$ is given by

$$\hat{N} = \sum_{j=1}^{n} c_j^\dagger c_j \,.$$

Obviously $[\hat{K}, \hat{N}] \neq 0$. We find the matrix representation of $\hat{K}$ and its eigenvalues and eigenvectors. We utilize the faithful matrix representation [6, 7, 8] for Fermi operators

$$c_k^\dagger = \overbrace{\sigma_z \otimes \cdots \otimes \sigma_z \otimes \left(\frac{1}{2}\sigma_+\right) \otimes I_2 \otimes \cdots \otimes I_2}^{n\text{-times}}$$

$$c_k = \underbrace{\sigma_z \otimes \cdots \otimes \sigma_z \otimes \left(\frac{1}{2}\sigma_-\right)}_{k\text{-th place}} \otimes I_2 \otimes \cdots \otimes I_2$$

where I_2 is the 2×2 identity matrix and σ_1, σ_2, σ_3 are the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with

$$\sigma_+ = \sigma_1 + i\sigma_2 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \qquad \sigma_- = \sigma_1 - i\sigma_2 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}.$$

We also calculate the unitary matrix $U(t) = \exp(-i\hat{H}t/\hbar)$ to solve the Schrödinger and Heisenberg equation of motion. Entangled and unentangled states can be found. As entanglement measure for the eigenvectors of Hamilton operator $\hat{K}$ we utilize the entanglement measure introduced by Wong and Christensen [10].

We also study the Lie algebra generated by $c_n^\dagger c_{n-1}^\dagger \cdots c_2^\dagger c_1^\dagger$ and $c_1 c_2 \cdots c_{n-1} c_n$ and the Lie algebra generated by the Hamilton operator $\hat{K}$ and the number operator $\hat{N}$.

2 Eigenvalue Problem for the Cases $n = 1$ and $n = 2$

For the case $n = 1$ we have the Hamilton operator

$$\hat{K} = \frac{\hat{H}}{\hbar\omega} = c^\dagger + c = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1$$

where the operators $c^\dagger$, c and $c^\dagger c$ are given by the 2×2 matrices

$$c^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad c = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \hat{N} = c^\dagger c = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and the basis is given by

$$c^\dagger|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Thus the Hamilton operator $\hat{H}$ acts in the Hilbert space $\mathbb{C}^2$. Obviously the eigenvalues of $\hat{K}$ are $+1$ and -1 with the corresponding normalized eigenvectors (Hadamard basis)

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

For the unitary operator $U(t) = \exp(-i\hat{H}t/\hbar)$ we obtain

$$U(t) = \exp(-i\hat{H}t/\hbar) = \begin{pmatrix} \cos(\omega t) & -i\sin(\omega t) \\ -i\sin(\omega t) & \cos(\omega t) \end{pmatrix}.$$

Consider now the case $n = 2$. The ordering of the four dimensional basis is $c_2^\dagger c_1^\dagger|0\rangle$, $c_2^\dagger|0\rangle$, $c_1^\dagger|0\rangle$, $|0\rangle$. Utilizing the matrix representation given above we have

$$c_1 = \frac{1}{2}\sigma_- \otimes I_2, \qquad c_2 = \sigma_3 \otimes \frac{1}{2}\sigma_-.$$

Thus we obtain the matrix representation

$$c_1c_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad c_2^\dagger c_1^\dagger = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Consequently

$$\hat{K} = c_2^\dagger c_1^\dagger + c_1c_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad [c_2^\dagger c_1^\dagger, c_1c_2] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The four eigenvalues of $\hat{K}$ are -1, 1, 0 (twice) with the corresponding bases for the eigenspaces

$$\left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \right\}, \quad \left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}, \quad \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

The first two eigenspaces have Bell states as basis and are fully entangled (except for the zero vector). The last eigenspace consists of unentangled and entangled vectors. For the number operator $\hat{N}$ we find

$$\hat{N} = c_1^\dagger c_1 + c_2^\dagger c_2 = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with eigenvalues 2, 1 (twice) and 0 and

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}, \quad \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}, \quad \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

as basis for the respective eigenspaces. For the unitary operator $U(t) = \exp(-i\hat{H}t/\hbar)$ we obtain

$$\exp(-i\hat{H}t/\hbar) = \begin{pmatrix} \cos(\omega t) & 0 & 0 & -i\sin(\omega t) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\sin(\omega t) & 0 & 0 & \cos(\omega t) \end{pmatrix}.$$

3 General Case

For arbitrary n and $n \geq 2$ the Hamilton operator $\hat{K}$ is given by the $2^n \times 2^n$ symmetric matrix over $\mathbb{R}$ with 1 at the entries $(1, 2^n)$ and $(2^n, 1)$ and otherwise 0, i.e.

$$\hat{K} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The commutator $[c_n^\dagger c_{n-1}^\dagger \cdots c_2^\dagger c_1^\dagger, c_1 c_2 \cdots c_{n-1} c_n]$ admits the matrix representation

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

This means we have a $2^n \times 2^n$ diagonal matrix with 1 at the entry $(1, 1)$ and -1 at the entry $(2^n, 2^n)$ and otherwise 0. The eigenvalues of $\hat{K}$ are given by 1, -1 and 0 ($2^n - 2$ times). The corresponding bases for the eigenspaces are

$$\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \right\}, \quad \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -1 \end{pmatrix} \right\}, \quad \left\{ \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \end{pmatrix} \right\}.$$

The first two eigenspaces consist of entangled vectors (except for the zero vector). The other $2^n - 2$ dimensional eigenspace includes entangled and unentangled vectors. For the unitary operator $U(t) = \exp(-i\hat{H}t/\hbar)$ we obtain

$$\exp(-i\hat{H}t/\hbar) = \begin{pmatrix} \cos(\omega t) & 0 & \dots & 0 & -i\sin(\omega t) \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ -i\sin(\omega t) & 0 & \dots & 0 & \cos(\omega t) \end{pmatrix} .$$

4 Lie Algebras

We are looking first at the Lie algebra generated by the two operators

$$c_n^\dagger c_{n-1}^\dagger \cdots c_2^\dagger c_1^\dagger, \qquad c_1 c_2 \cdots c_{n-1} c_n .$$

Consider first the case $n = 1$. Since $[c^\dagger, c] = 2c^\dagger c - I$ and

$$[c^\dagger, 2c^\dagger c - I] = -2c^\dagger, \quad [c, 2c^\dagger c - I] = 2c$$

we find a three-dimensional simple Lie algebra with the basis

$$c^\dagger, \quad c, \quad c^\dagger c - \frac{I}{2} .$$

Thus we have a basis of the simple Lie algebra $s\ell(2, \mathbb{R})$. The matrix representation is

$$c^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad c^\dagger c - \frac{I}{2} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Consider now the case with $n = 2$ and the Lie algebra generated by $c_2^\dagger c_1^\dagger$ and $c_1 c_2$. We have

$$[c_2^\dagger c_1^\dagger, c_1 c_2] = c_1^\dagger c_1 + c_2^\dagger c_2 - I$$

Next we obtain the commutators

$$[c_2^\dagger c_1^\dagger, c_1^\dagger c_1 + c_2^\dagger c_2 - I] = 2c_1^\dagger c_2^\dagger, \qquad [c_1 c_2, c_1^\dagger c_1 + c_2^\dagger c_2 - I] = 2c_1 c_2 .$$

Thus we have a simple three-dimensional Lie algebra with the basis $c_2^\dagger c_1^\dagger$, $c_1 c_2$, $c_1^\dagger c_1 + c_2^\dagger c_2 - I$. The Lie algebra is isomorphic to $s\ell(2, \mathbb{R})$. The matrix representation given by the diagonal matrix

$$c_1^\dagger c_1 + c_2^\dagger c_2 - I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and

$$c_1c_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad c_2^\dagger c_1^\dagger = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Consider now the case $n = 3$ and the Lie algebra generated by the operators $c_3^\dagger c_2^\dagger c_1^\dagger$ and $c_1c_2c_3$. We obtain the commutator

$$[c_3^\dagger c_2^\dagger c_1^\dagger, c_1c_2c_3] = 2c_3^\dagger c_2^\dagger c_1^\dagger c_1c_2c_3 - c_2^\dagger c_1^\dagger c_1c_2 - c_3^\dagger c_2^\dagger c_2c_3 - c_3^\dagger c_1^\dagger c_1c_3 + c_1^\dagger c_1 + c_2^\dagger c_2 + c_3^\dagger c_3 - I\,.$$

Next we find

$$[c_1c_2c_3, [c_3^\dagger c_2^\dagger c_1^\dagger, c_1c_2c_3]] = 2c_1c_2c_3$$
$$[c_3^\dagger c_2^\dagger c_1^\dagger, [c_3^\dagger c_2^\dagger c_1^\dagger, c_1c_2c_3]] = -2c_3^\dagger c_2^\dagger c_1^\dagger\,.$$

Thus the three operators $c_1c_2c_3$, $c_3^\dagger c_2^\dagger c_1^\dagger$, $[c_3^\dagger c_2^\dagger c_1^\dagger, c_1c_2c_3]$ form a basis of a three-dimensional Lie algebra which is isomorphic to $s\ell(2,\mathbb{R})$. The matrix representation of $[c_3^\dagger c_2^\dagger c_1^\dagger, c_1c_2c_3]$ is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

For arbitrary n we have

$$[c_1 \cdots c_n, [c_n^\dagger \cdots c_1^\dagger, c_1 \cdots c_n]] = 2c_1 \cdots c_n$$
$$[c_n^\dagger \cdots c_1^\dagger, [c_n^\dagger \cdots c_1^\dagger, c_1 \cdots c_n]] = -2c_n^\dagger \cdots c_1^\dagger\,.$$

Thus for arbitrary n the three operators

$$c_n^\dagger c_{n-1}^\dagger \cdots c_2^\dagger c_1^\dagger, \qquad c_1c_2 \cdots c_{n-1}c_n, \qquad [c_n^\dagger c_{n-1}^\dagger \cdots c_2^\dagger c_1^\dagger, c_1c_2 \cdots c_{n-1}c_n]$$

form a basis of a three dimensional Lie algebra which is isomorphic to $s\ell(2,\mathbb{R})$.

Next we study the Lie algebra generated by the Hamilton operator $\hat{K}$ and the number operator $\hat{N}$. Let $n = 1$. We find for the commutators

$$[\hat{K}, \hat{N}] = c - c^\dagger, \quad [\hat{K}, c - c^\dagger] = 4c^\dagger c - 2I = 4\hat{N} - 2I, \quad [\hat{N}, c - c^\dagger] = -c^\dagger - c = -\hat{K}\,.$$

Thus we have a four-dimensional non-commutative Lie algebra with a basis given by $\hat{K}$, $\hat{N}$, $c - c^\dagger$, I. Owing to the operator I the Lie algebra is not semisimple. Utilizing the matrix representation we have

$$\hat{K} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1, \quad \hat{N} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad c - c^\dagger = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -i\sigma_1, \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For $n = 2$ we have the commutators

$$\begin{aligned}
[\hat{K}, \hat{N}] &= 2(c_1c_2 - c_2^\dagger c_1^\dagger) \\
[\hat{K}, [\hat{K}, \hat{N}]] &= 4(\hat{N} - I) \\
[\hat{N}, [\hat{K}, \hat{N}]] &= -4\hat{K} \\
[\hat{K}, [\hat{K}, [\hat{K}, \hat{N}]]] &= 4[\hat{K}, \hat{N}] \\
[\hat{N}, [\hat{K}, [\hat{K}, \hat{N}]]] &= 0 \\
[[\hat{K}, \hat{N}], [\hat{K}, [\hat{K}, \hat{N}]]] &= 16\hat{K}\,.
\end{aligned}$$

Thus the operators $\hat{K}$, $\hat{N}$, $c_1c_2 - c_2^\dagger c_1^\dagger$, I form a basis of the four dimensional Lie algebra which is not semisimple owing to I. The matrix representation is

$$\hat{K} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{N} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad c_1c_2 - c_2^\dagger c_1^\dagger = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

For $n = 3$ we have the commutators

$$\begin{aligned}
[\hat{K}, \hat{N}] &= 3(c_1c_2c_3 - c_3^\dagger c_2^\dagger c_1^\dagger) \\
[\hat{K}, [\hat{K}, \hat{N}]] &= 6[c_3^\dagger c_2^\dagger c_1^\dagger, c_1c_2c_3] \\
[\hat{N}, [\hat{K}, \hat{N}]] &= -9\hat{K} \\
[\hat{K}, [\hat{K}, [\hat{K}, \hat{N}]]] &= 4[\hat{K}, \hat{N}] \\
[\hat{N}, [\hat{K}, [\hat{K}, \hat{N}]]] &= 0 \\
[[\hat{K}, \hat{N}], [\hat{K}, [\hat{K}, \hat{N}]]] &= 36\hat{K}\,.
\end{aligned}$$

Thus we have a four dimensional Lie algebra given by the operators $\hat{K}$, $\hat{N}$, $[\hat{K}, \hat{N}]$, $[\hat{K}, [\hat{K}, \hat{N}]]$. The Lie algebra is not semisimple.

For general n we have

$$\begin{aligned}
[\hat{K}, \hat{N}] &= n(c_1 \cdots c_n - c_n^\dagger \cdots c_1^\dagger) \\
[\hat{K}, [\hat{K}, \hat{N}]] &= 2n[c_n^\dagger \cdots c_1^\dagger, c_1 \cdots c_n] \\
[\hat{N}, [\hat{K}, \hat{N}]] &= -n^2\hat{K} \\
[\hat{K}, [\hat{K}, [\hat{K}, \hat{N}]]] &= 4[\hat{K}, \hat{N}] \\
[\hat{N}, [\hat{K}, [\hat{K}, \hat{N}]]] &= 0 \\
[[\hat{K}, \hat{N}], [\hat{K}, [\hat{K}, \hat{N}]]] &= 4n^2\hat{K}\,.
\end{aligned}$$

Thus we find that the four operators $\hat{K}$, $\hat{N}$, $[\hat{K}, \hat{N}]$, $[\hat{K}, [\hat{K}, \hat{N}]]$ provide a basis of a four dimensional Lie algebra which is not semisimple.

5 Entanglement

An n-tangle [8, 9, 10] can be defined for the finite dimensional Hilbert space $\mathcal{H} = \mathbb{C}^{2^n}$, with $n = 3$ or n even. Consider the finite-dimensional Hilbert space $\mathcal{H} = \mathbb{C}^{2^n}$ and the normalized states

$$|\psi\rangle = \sum_{j_1,j_2,\ldots,j_n=0}^{1} c_{j_1,j_2,\ldots,j_n} |j_1\rangle \otimes |j_2\rangle \otimes \cdots \otimes |j_n\rangle$$

in this Hilbert space. Here $|0\rangle$, $|1\rangle$ denotes the standard basis. Let ϵ_{jk} $(j,k = 0,1)$ be defined by $\epsilon_{00} = \epsilon_{11} = 0$, $\epsilon_{01} = 1$, $\epsilon_{10} = -1$. Let n be even or $n = 3$. Then an n-tangle can be introduced by

$$\tau_{1\ldots n} = 2 \Bigg| \sum_{\substack{\alpha_1,\ldots,\alpha_n=0 \\ \vdots \\ \delta_1,\ldots,\delta_n=0}}^{1} c_{\alpha_1\ldots\alpha_n} c_{\beta_1\ldots\beta_n} c_{\gamma_1\ldots\gamma_n} c_{\delta_1\ldots\delta_n} \times \epsilon_{\alpha_1\beta_1}\epsilon_{\alpha_2\beta_2}\cdots\epsilon_{\alpha_{n-1}\beta_{n-1}}\epsilon_{\gamma_1\delta_1}\epsilon_{\gamma_2\delta_2}\cdots\epsilon_{\gamma_{n-1}\delta_{n-1}}\epsilon_{\alpha_n\gamma_n}\epsilon_{\beta_n\delta_n} \Bigg| .$$

This includes the definition for the 3-tangle with $n = 3$.

Consider now the eigenvectors of the Hamilton operator $\hat{K}$ with $n \geq 2$. Then the eigenvectors belonging to -1 and $+1$ are fully entangled and include part of the Bell basis. The eigenspace belonging to the eigenvalue 0 consists of both entangled and unentangled vectors.

6 Conclusion

We have studied a Fermi Hamilton operator. If the eigenvalues are degenerate then by linear combinations we can construct entangled states from unentangled states. A computer algebra program written in SymbolicC++[11] for the manipulation of the Fermi operators is available from the authors.

The model described above has a straightforward extension to the Fermi Hamilton operator with spin

$$\hat{K} = \frac{\hat{H}}{\hbar\omega} = c_{n\uparrow}^\dagger c_{n-1\uparrow}^\dagger \cdots c_{2\uparrow}^\dagger c_{1\uparrow}^\dagger + c_{n\downarrow}^\dagger c_{n-1\downarrow}^\dagger \cdots c_{2\downarrow}^\dagger c_{1\downarrow}^\dagger + c_{1\uparrow}c_{2\uparrow}\cdots c_{n-1\uparrow}c_{n\uparrow} + c_{1\downarrow}c_{2\downarrow}\cdots c_{n-1\downarrow}c_{n\downarrow}$$

with the number operator $\hat{N}$ and spin operator $\hat{S}_z$ given by

$$\hat{N} = \sum_{j=1}^{n}(c_{j\uparrow}^\dagger c_{j\uparrow} + c_{j\downarrow}^\dagger c_{j\downarrow}), \qquad \hat{S}_z = \frac{1}{2}\left(\sum_{j=1}^{n}(c_{j\uparrow}^\dagger c_{j\uparrow} - c_{j\downarrow}^\dagger c_{j\downarrow})\right)$$

where $[\hat{K}, \hat{N}] \neq 0$, $[\hat{K}, \hat{S}_z] \neq 0$, $[\hat{N}, \hat{S}_z] = 0$. Here the matrix representation is given by [6, 7, 8, 9]

$$
\begin{aligned}
& \qquad\qquad\qquad k\text{-th place} \\
c^\dagger_{k\uparrow} &= \underbrace{\sigma_z \otimes \cdots \otimes \sigma_z \otimes \left(\frac{1}{2}\sigma_+\right) \otimes I_2 \otimes \cdots \otimes I_2}_{2n \text{ times}} \\
c^\dagger_{k\downarrow} &= \sigma_z \otimes \cdots \otimes \sigma_z \otimes \left(\frac{1}{2}\sigma_+\right) \otimes I_2 \otimes \cdots \otimes I_2 \\
& \qquad\qquad\qquad (k+n)\text{-th place}
\end{aligned}
$$

where $k = 1, \ldots, n$. Whereas the Fermi system discussed above provides a three energy level system for $n = 2$ with eigenvalues 1, 0, -1 including the spin provides five energy levels with eigenvalues 2, 1, 0, -1, -2 for $n = 2$.

The model discussed above can easily be extended to Majorana fermions on a lattice [12, 13, 14, 15]. Given a set of n (spin-less) Fermi creation and annihilations operators $c_j^\dagger$, c_j ($j = 1, \ldots, n$) we can define the set of $2n$ (real) Majorana fermion operators on a lattice γ_{j1}, γ_{j2} ($j = 1, \ldots, n$) as

$$
c_j = \frac{1}{2}(\gamma_{j1} + i\gamma_{j2}), \qquad c_j^\dagger = \frac{1}{2}(\gamma_{j1} - i\gamma_{j2})
$$

where $\gamma_{j1}^* = \gamma_{j1}$, $\gamma_{j2}^* = \gamma_{j2}$ and $\gamma_{j1}^2 = \gamma_{j2}^2 = I$. It follows that

$$
\gamma_{j1} = c_j^\dagger + c_j, \qquad \gamma_{j2} = i(c_j^\dagger - c_j)
$$

with

$$
[\gamma_{j1}, \gamma_{j2}]_+ = 0, \quad [\gamma_{j1}, \gamma_{j1}]_+ = 2I, \quad [\gamma_{j2}, \gamma_{j2}]_+ = 2I
$$

and $[\gamma_{j\ell}, \gamma_{k\ell'}]_+ = 0$ for $j \neq k$.

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